Homework 2

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R Markdown

This is an R Markdown document. Markdown is a simple formatting syntax for authoring HTML, PDF, and MS Word documents. For more details on using R Markdown see http://rmarkdown.rstudio.com.

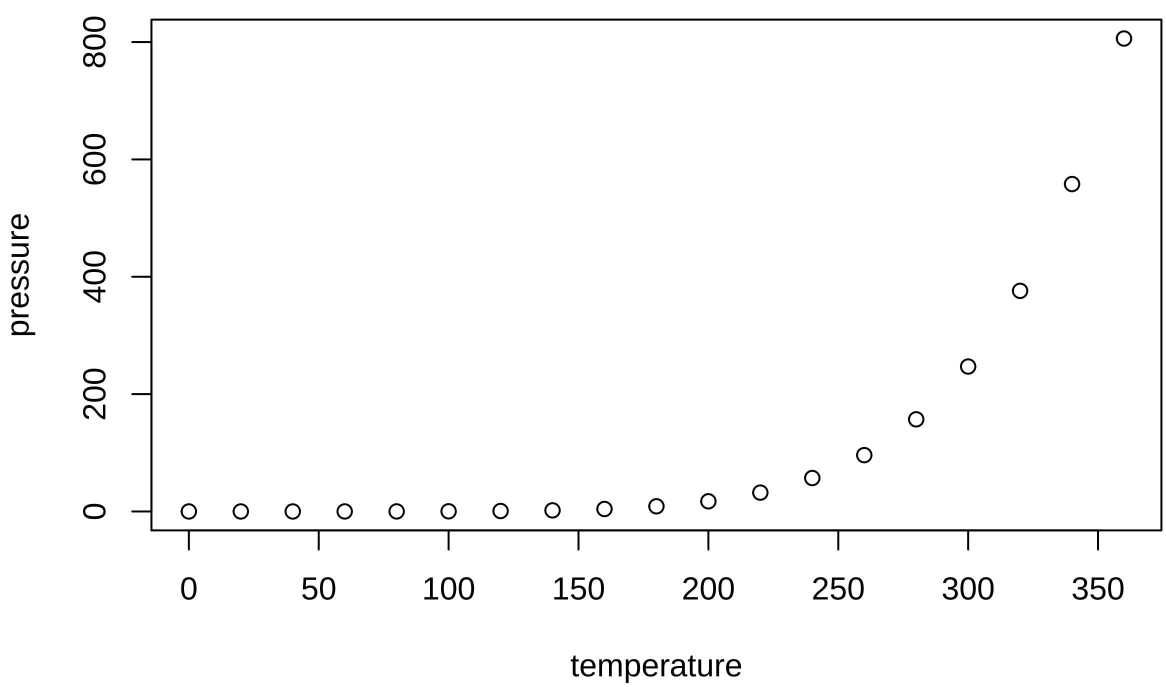
When you click the **Knit** button a document will be generated that includes both content as well as the output of any embedded R code chunks within the document. You can embed an R code chunk like this:

```r
summary(cars)
```

```
##      speed           dist       
##  Min.   : 4.0   Min.   :  2.00  
##  1st Qu.:12.0   1st Qu.: 26.00  
##  Median :15.0   Median : 36.00  
##  Mean   :15.4   Mean   : 42.98  
##  3rd Qu.:19.0   3rd Qu.: 56.00  
##  Max.   :25.0   Max.   :120.00  
```

Including Plots

You can also embed plots, for example:



Note that the `echo = FALSE` parameter was added to the code chunk to prevent printing of the R code that generated the plot.

Problem 1:

(1) A discrete random variable X has a probability mass function of the form $P(X = x) = \frac{k}{2^x}$ for $x = 1, 2, 3$ and zero otherwise, find k.

We know that the $\sum_{x=1}^{3} \frac{k}{2^x} = 1$. Therefore, we have that:

$$\begin{aligned}\sum_{x=1}^{3} \frac{k}{2^x} &= \frac{k}{2^1} + \frac{k}{2^2} + \frac{k}{2^3} \\ &= \frac{k}{2} + \frac{k}{4} + \frac{k}{8} \\ &= \frac{7k}{8} \\ \Rightarrow k &= \frac{8}{7} \\ &\square\end{aligned}$$

(2) Can a function of the form $f(x) = c(2^{-x} - 0.5)$ for $x = 0, 1, 2$ and zero otherwise be a probability mass function of a random variable?

$f(0) = c(2^{-0} - 0.5) = .50c$
$f(1) = c(2^{-1} - 0.5) = 0$
$f(2) = c(2^{-2} - 0.5) = -.25c$
$f(0) + f(1) + f(2) = .25$
$\Rightarrow c = 4$

However, we know that $f(2) = 4(2^{-2} - 0.5) = -1$ Which is a contradiction since for a pdf $f(x) \geq 0$. Thus $f(x)$ is not a pdf $\square$

Problem 2: A function of the form $f(t) = ct^{-c-1}I\{t > 1\}$ for $t \in (-\infty, \infty)$.

(1) If $f(t)$ is a probability density function, find the value of c.

We use the definition of a pdf to show that:

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)dt &= 1 \\ \int_{-\infty}^{\infty} f(t)dt &= \int_{-\infty}^{\infty} ct^{-c-1}I\{t > 1\}dt \\ &= \int_{1}^{\infty} ct^{-c-1}dt \\ &= -t^{-c}\Big|_{1}^{\infty} \\ &= \frac{1}{\infty^c} + 1^{-c} \\ \Rightarrow c &> 0\end{aligned}$$

We know this since if $c = 0$ then we would get that $f(t) = 0 \Rightarrow \int_{-\infty}^{\infty} f(t)tx = \int_{-\infty}^{\infty} 0dt = 0$ which is a contradiction of the definition of a pdf.

Furthermore if we have $c < 0$ then we would get that $\frac{1}{\infty^c} + 1^{-c} = \infty$, which also contradicts the definition of a pdf and that $\int_{-\infty}^{\infty} f(t)dt = 1\square$.

(2) Find the corresponding cumulative distribution function of $f(t)$ in (1).

$$\begin{aligned}
P(T \leq t) &= F_T(t) \\
&= \int_{-\infty}^{t} f_T(x)dx \\
&= \int_{-\infty}^{t} cx^{-c-1} I\{x > 1\}dx \\
&= \int_{1}^{t} cx^{-c-1}dx \\
&= -x^{-c}\Big|_1^t \\
&= -t^{-c} + 1^{-c} \\
&= (1 - \frac{1}{t^c})I\{t > 1\}
\end{aligned}$$

Problem 3: Suppose $f(t)$ and $g(t)$ for $t \in (-\infty, \infty)$ are probability density functions. Let $a \geq 0$ and $b \geq 0$ are two fixed constants satisfying $a + b = 1$. Prove that $af(t) + bg(t)$ is also a probability density function for $t \in (-\infty, \infty)$.

To show that $af(t) + bg(t)$ is a pdf we prove that $af(t) + bg(t) \geq 0, \forall x$ and $\int_{-\infty}^{\infty} af(t) + bg(t)dt = 1$.

1. We first show that $af(t) + bg(t) \geq 0$:

Assume that there exist a t such that $af(t) + bg(t) < 0$. This implies that at least one of the two terms is the function are negative. From this we know that if $af(t) < 0$ either $a < 0$ or $f(t) < 0$ which is a contradiction since we know that $f(t) \geq 0$ (by the definition of a pdf) and $a > 0$ by the statement of the problem. If $bg(t) < 0$ then either $b < 0$ or $g(t) < 0$ which is also a contradiction, since we know that $g(t) \geq 0$ (by the definition of a pdf) and $b > 0$ by the statement of the problem. Therefore by contradiction, we know that $af(t) + bg(t) \geq 0$

2. We show that $\int_{-\infty}^{\infty}(af(t) + bg(t))dx = 1$

$$\begin{aligned}
\int_{-\infty}^{\infty}(af(t) + bg(t))dt &= \int_{-\infty}^{\infty} af(t)dt + \int_{-\infty}^{\infty} bg(t)dx \\
&= a\int_{-\infty}^{\infty} f(t)dt + b\int_{-\infty}^{\infty} g(t)dx \\
&= a(1) + b(1), \text{since the Defintion of pdf we know that} \int_{-\infty}^{\infty} f(t)dt = 1, \int_{-\infty}^{\infty} g(t)dt = 1 \\
&= a + b \\
&= 1
\end{aligned}$$

From this we know that the definition of a pdf holds for $af(t) + bg(t)$. $\square$